

A Matrix Model for QCD: QCD Colour is Mixed

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Abstract

We use general arguments to show that coloured QCD states when restricted to gauge invariant local observables are mixed. This result has important implications for confinement: a pure colourless state can never evolve into two coloured states by unitary evolution. Furthermore, the mean energy in such a mixed coloured state is infinite. Our arguments are confirmed in a matrix model for QCD that we have developed using the work of Narasimhan and Ramadas [3] and Singer [2]. This model, a $(0+1)$ -dimensional quantum mechanical model for gluons free of divergences and capturing important topological aspects of QCD, is adapted to analytical and numerical work. It is also suitable to work on large N QCD. As applications, we show that the gluon spectrum is gapped and also estimate some low-lying levels for $N = 2$ and 3 (colors).

Incidentally the considerations here are generic and apply to any non-abelian gauge theory.

1 Introduction

The understanding of physical states in QCD is of fundamental importance. Conjectures regarding quark confinement and chiral symmetry breaking are based on speculations about their nature. It is also important for a non-perturbative formulation of QCD.

Gribov [1] showed many years ago that the Coulomb gauge in QCD does not fully fix the gauge and is inadequate for a non-perturbative formulation of QCD. Later, Singer [2] and Narasimhan and Ramadas [3] proved that the Gribov problem cannot be resolved by choosing another gauge condition since the gauge bundle on the QCD configuration space is twisted.

In this paper, we argue that as a consequence of the above twisted nature of the QCD bundle, coloured states restricted to the algebra of local observables are necessarily mixed: they carry entropy. This argument is confirmed in a matrix model for gluons we also propose here. This model is $0 + 1$ dimensional and free of the technical problems of quantum field theory.

The matrix model, being a quantum mechanical model of 8×8 real matrices, for $N = 3$ colours, and capturing certain essential topological aspects of QCD offers a new approach to QCD calculations. It is also suitable for the study of 't Hooft's large N limit. As an explicit

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illustration of the power of our approach, we show that the gluon spectrum has a gap in our model. In lattice calculations this is taken as a signal for confinement.

For $N = 2$ and 3 , we also use simple variational calculations to estimate low-lying glueball masses. Detailed numerical work is on progress.

Just as in a soliton model, it is necessary to quantise the excitations around our matrix model solutions in a full quantum field theory. In this connection, we note that the matrix model contains the vacuum sector where the gluon potential is gauge equivalent to the zero field. We also indicate how to construct multiparticle levels for our gluon levels adapting standard techniques in soliton physics [4].

In a paper under preparation, we will argue that QCD has different phases, and also calculate the glueball spectrum in these phases. The Dirac operator in the matrix model approach will also be discussed.

2 The Gauge Bundle in QCD

Let $A_i = A_i^\alpha(\lambda_\alpha/2)$, with $i = 1, 2, 3$ and λ_α being Gell-Mann matrices, denote the QCD vector potentials (in our convention, $D_\mu = \partial_\mu + A_\mu$, with $A_\mu^\dagger = -A_\mu$) in the temporal $A_0 = 0$ gauge. Its gluon configuration space Q is based on the space $\mathcal{A} = \{A = (A_1, A_2, A_3)\}$ of their connections. The QCD gauge group \mathcal{G} is the group $\{u\}$ of maps from \mathbb{R}^3 to $SU(3)$ with the asymptotic condition (time argument is suppressed)

$$u(\vec{x}) \xrightarrow{|\vec{x}| \rightarrow \infty} u_\infty \in SU(3). \quad (2.1)$$

(See also the Sky group in this respect [5].) The group \mathcal{G} acts on \mathcal{A} according to

$$u \cdot A_i \mapsto u A_i u^{-1} + u \partial_i u^{-1}. \quad (2.2)$$

There are two normal subgroups of \mathcal{G} of importance here,

$$\mathcal{G}^\infty = \{u \in \mathcal{G}, \quad u(x) \rightarrow \text{the identity } e \text{ of } SU(3) \text{ as } |\vec{x}| \rightarrow \infty\}, \quad (2.3)$$

$$\mathcal{G}_0^\infty = \text{connected component of } \mathcal{G}^\infty. \quad (2.4)$$

As discussed elsewhere [4, 5], the Gauss law generates \mathcal{G}_0^∞ which therefore acts trivially on the physical quantum states.

The group \mathcal{G}_0^∞ is normal in \mathcal{G}^∞ and

$$\mathcal{G}^\infty / \mathcal{G}_0^\infty = \mathbb{Z}. \quad (2.5)$$

Its representations $\mathbb{Z} \ni n \rightarrow e^{in\theta}$ characterise the θ -states of QCD.

The colour group is

$$\mathcal{G} / \mathcal{G}^\infty = SU(3). \quad (2.6)$$

All observables commute with the Gauss law, that is, \mathcal{G}_0^∞ . In quantum physics, observables are also *local* [6], that is, they are obtained by smearing standard quantum fields with test functions with supports in compact spacetime regions. In the canonical formalism, that means that local observables are obtained from smeared quantum fields over a compact¹ spatial region K . Call

¹Instead of a fixed time slice, if one considers a time average of a field $\varphi(\vec{x}, t)$ over given arbitrarily small, but finite time slices, matrix elements of fields become smooth functions on \mathbb{R}^3 [7].

such a field $\varphi(K)$. The action of $u \in \mathcal{G}$ on $\varphi(K)$ depends only on the restriction $u|_K$ of u to K . But $u|_K$ can be smoothly extended beyond K to a gauge transformation $u' \in \mathcal{G}_0^\infty$. There are many ways of doing so and for each u' , by Gauss law, if $\varphi(K)$ is an observable, then

$$u'\varphi(K) = \varphi(K)u'. \quad (2.7)$$

Hence,

$$u\varphi(K) = \varphi(K)u, \quad (2.8)$$

so that *all local observables commute with elements of \mathcal{G}* .

The configuration space Q for local observables is thus associated with $Q = \mathcal{A}/\mathcal{G}$ and not $\mathcal{A}/\mathcal{G}_0^\infty$ as naive considerations using the Gauss law would suggest.

Quantum vector states Ψ instead can be built from maps from \mathcal{A} to \mathbb{C} which are annihilated by the Gauss law:

$$\begin{aligned} \Psi : \mathcal{A} &\rightarrow \mathbb{C}, & \Psi(A) &\in \mathbb{C}, \\ (u \cdot \Psi)(A) &= \Psi(A), & \text{if } u &\in \mathcal{G}_0^\infty. \end{aligned} \quad (2.9)$$

Hence wave functions are sections of vector bundles built on $\mathcal{A}/\mathcal{G}_0^\infty$. It follows that we have the fibre bundle structure

$$\pi : \mathcal{A}/\mathcal{G}_0^\infty \rightarrow \mathcal{A}/\mathcal{G}, \quad (2.10)$$

for the group

$$\mathcal{G}/\mathcal{G}_0^\infty = SU(3) \times \mathbb{Z}. \quad (2.11)$$

Any function on $Q = \mathcal{A}/\mathcal{G}$ is invariant under gauge transformations, and is hence a colour singlet.

The bundle (2.10) is twisted. Otherwise we would conclude that $\mathcal{A}/\mathcal{G}_0^\infty = \mathcal{A}/\mathcal{G} \times (SU(3) \times \mathbb{Z})$, which is false since $\mathcal{A}/\mathcal{G}_0^\infty$ is connected. This last statement follows from the fact that \mathcal{A} itself is connected.

This argument however must be sharpened since $\mathcal{G}/\mathcal{G}_0^\infty$ does not act freely on $\mathcal{A}/\mathcal{G}_0^\infty$. Indeed, an element of $\mathcal{A}/\mathcal{G}_0^\infty$ is

$$\mathcal{G}_0^\infty A := \langle u \cdot A, u \in \mathcal{G}_0^\infty \rangle. \quad (2.12)$$

The action of $h\mathcal{G}_0^\infty \in \mathcal{G}/\mathcal{G}_0^\infty$ on this element is

$$\mathcal{G}_0^\infty A \rightarrow (h\mathcal{G}_0^\infty) \cdot \mathcal{G}_0^\infty A = \mathcal{G}_0^\infty (h \cdot A), \quad (2.13)$$

since \mathcal{G}_0^∞ is normal in \mathcal{G} . To see explicitly that the action is not free, choose $A = \lambda_8 a_8$ and $h \in SU(2) \subset SU(3)$ with Lie algebra basis $\lambda_1, \lambda_2, \lambda_3$ to find that $h\mathcal{G}_0^\infty$ leaves $\mathcal{G}_0^\infty A$ invariant. Hence $\mathcal{G}/\mathcal{G}_0^\infty$ does not act freely on $\mathcal{A}/\mathcal{G}_0^\infty$.

The centre \mathbb{Z}_3 of $SU(3)$ leaves all vector potentials A invariant, so we can change \mathcal{G} to $Ad\mathcal{G} = \mathcal{G}/\mathbb{Z}_3$, and correspondingly define $Ad\mathcal{G}_0^\infty$ and $Ad\mathcal{G}_0^\infty$.

We next consider *generic* connections \mathcal{A}_0 with holonomy at any point $\vec{x}_0 \in \mathbb{R}^3$ being $Ad\,SU(3)$. Then the above Ad groups act freely on \mathcal{A}_0 [2, 3], so that we obtain the principal fibre bundle

$$\begin{aligned} \pi : \mathcal{A}_0/Ad\mathcal{G}_0^\infty &\rightarrow (\mathcal{A}_0/Ad\mathcal{G}_0^\infty)/(Ad\mathcal{G}/Ad\mathcal{G}_0^\infty), \\ Ad\mathcal{G}/Ad\mathcal{G}_0^\infty &\simeq Ad\,SU(3) \times \mathbb{Z}. \end{aligned} \quad (2.14)$$

Previous authors [2, 3] had shown that this bundle is twisted, that is, non-trivial,

$$\mathcal{A}_0/Ad\mathcal{G}_0^\infty \neq (\mathcal{A}_0/Ad\mathcal{G}_0^\infty)/(Ad\mathcal{G}/Ad\mathcal{G}_0^\infty) \times (AdSU(3) \times \mathbb{Z}). \quad (2.15)$$

A quick proof is due to Singer, see his Theorem 2 in [2]. He starts with the fact that $\pi_j(\mathcal{A}_0) = \{0\}$, for any $j \in \mathbb{N}$ where $\pi_j(\mathcal{A}_0)$ is the j th homotopy group of \mathcal{A}_0 . In particular, since $\pi_0(\mathcal{A}_0) = \{0\}$, then $\pi_0(\text{LHS of (2.15)}) = \{0\}$. But on the RHS of (2.15), we have that $\pi_0(\mathbb{Z}) = \mathbb{Z}$. Also, $\pi_1(\text{LHS of (2.15)}) = \{0\}$, since $\pi_0(Ad\mathcal{G}_0^\infty) = \{0\}$, while on the RHS we have $\pi_1(AdSU(3)) = \mathbb{Z}_3$. Thus, since the LHS and RHS of (2.15) have different homotopy groups, we conclude that they cannot be equal. For a related discussion of the relevant cohomologies, see [10].

The non-generic connections lead to some sort of boundary points. More precisely, these “boundary points” give a “stratified” manifold [11].

A similar situation is already known to happen in a different context. Recall the treatment of N identical particles on \mathbb{R}^d [4]. In this case, the bundle space is

$$\overline{Q}_N = \{(x_1, \dots, x_N), \quad x_i \in \mathbb{R}^d\}, \quad (2.16)$$

whereas the configuration space is

$$\overline{Q}_N/S_N = \{[x_1, \dots, x_N]\}, \quad (2.17)$$

where S_N acts by permutations of x_i ’s and $[x_1, \dots, x_N]$ is an *unordered* set

$$[x_1, \dots, x_i, \dots, x_j, \dots, x_N] = [x_1, \dots, x_j, \dots, x_i, \dots, x_N]. \quad (2.18)$$

But if $x_i = x_j$, for some $i, j \in \{1, \dots, N\}$, then (x_1, x_2, \dots, x_N) is invariant under the transformation $x_i \leftrightarrow x_j$, so that the action of S_N on \overline{Q}_N is not free. Hence to get a genuine fibre bundle, we exclude coincidence of any two points and work with

$$\overline{Q}_N^0 = \{(x_1, \dots, x_N), \quad x_i \neq x_j \text{ if } i \neq j\}. \quad (2.19)$$

Then

$$S_N : \overline{Q}_N^0 \rightarrow Q_N := \overline{Q}_N^0/S_N \quad (2.20)$$

gives a principle fibre bundle. This bundle is also twisted.

Given an operator like the Laplacian Δ on \overline{Q}_N^0 , the points of \overline{Q}_N with $x_i = x_j$ turn up as “boundary points” where suitable boundary conditions have to be imposed.

Likewise, the non-generic connections may have to be treated by suitable conditions in an appropriate setting. They are conjectured to lead to different phases of QCD. We will take up these issues in another paper. But we will not encounter the need for such conditions in the approach taken here.

Since the bundle (2.10) is twisted, previous works [2, 3] infer that $SU(N)$ (or $U(N)$) gauge theories do not admit global gauge conditions.

In conclusion, we have the twisted bundle (2.10) in QCD. Wave functions are functions on $\mathcal{A}_0/Ad\mathcal{G}_0^\infty$ which under $AdSU(3) \times \mathbb{Z}$ transform by one of its unitary irreducible representations (UIR’s). Local observables instead are colour singlets.

3 How Mixed States Arise

The UIR's $n \rightarrow e^{in\theta}$ of \mathbb{Z} lead to θ -states. We will remark on them in Section 8.

For now, we focus on $SU(3)$. Hence consider the wave functions

$$\begin{aligned} & |[a_0]; \rho, \lambda), \quad a_0 \in \mathcal{A}_0, \\ & a_0, a'_0 \in [a_0] \Leftrightarrow a'_0 = u \cdot a_0, \quad u \in \mathcal{G}_0^\infty, \end{aligned} \quad (3.1)$$

transforming as the component λ of the UIR ρ of $SU(3)$

$$|[u \cdot a_0]; \rho, \lambda) = |[a_0]; \rho, \lambda') D_{\lambda' \lambda}^\rho(u), \quad u \in SU(3). \quad (3.2)$$

The corresponding density matrix, from which the state on the space of observables is defined, is

$$\omega([a_0]; \rho, \lambda) = |[a_0]; \rho, \lambda)([a_0]; \rho, \lambda|, \quad (3.3)$$

where we assume for simplicity that the kets are normalised to 1 in a suitable scalar product. (Actually, we must really consider wave packets in $[a_0]$).

The observable algebra we work with is the algebra \mathcal{C} of colour singlet operators. They are associated with \mathcal{A}/\mathcal{G} . \mathcal{C} contains $\mathbf{1}$. We assume that it is a C^* -algebra, though this point does not enter the formal considerations here. The algebra \mathcal{C}_{Loc} , the algebra of local observables, is a subalgebra of \mathcal{C} , so that a mixed state on \mathcal{C} remains mixed when restricted to \mathcal{C}_{Loc} . In what follows, we work with \mathcal{C} itself.

If $b \in \mathcal{C}$, then its mean value in the state (3.3) is

$$\omega([a_0]; \rho, \lambda)(b) = ([a_0]; \rho, \lambda| b |[a_0]; \rho, \lambda). \quad (3.4)$$

If ρ is the colour singlet representation, the state (3.3) restricted to \mathcal{C} is pure. But that is not the case if ρ is a non-trivial $SU(3)$ UIR. We now show this result using the GNS construction. The argument is modelled on our previous work on ethylene [8].

Suppose now that ρ is a non-trivial $SU(3)$ UIR. We introduce the vector states

$$|b\rangle, \quad b \in \mathcal{C}, \quad (3.5)$$

and the inner product

$$\langle b'|b\rangle = \omega([a_0]; \rho, \lambda)(b'^* b). \quad (3.6)$$

We emphasize that the GNS inner product $\langle \cdot | \cdot \rangle$ is different from $(\cdot | \cdot)$.

Consider the projector

$$\mathbb{P} = \sum_{\lambda} |[a_0]; \rho, \lambda)([a_0]; \rho, \lambda| \equiv \sum_{\lambda} \mathbb{P}_{\lambda}, \quad (3.7)$$

which is a colour singlet and hence is an element of \mathcal{C} . Further if

$$0 \neq n \in \mathcal{C} \left(|1\rangle \langle 1| - \mathbb{P} \right) := \mathcal{N}, \quad \text{the Gelfan'd ideal}, \quad (3.8)$$

then n is a null vector, that is,

$$\langle n | n \rangle = 0. \quad (3.9)$$

Thus we introduce the equivalence classes

$$\tilde{b} = \{b + n, \quad n \in \mathcal{N}\}, \quad (3.10)$$

and the vector $|\tilde{b}\rangle$, so that

$$\langle \tilde{b}' | \tilde{b} \rangle = \omega([a_0]; \rho, \lambda)(b'^* b). \quad (3.11)$$

There are no non-zero null vectors among $|\tilde{b}\rangle$. The completion of $\{|\tilde{b}\rangle\}$ in the scalar product (3.6) gives the Hilbert space \mathcal{H}_{GNS} .

The representation σ of \mathcal{C} on \mathcal{H}_{GNS} is

$$\sigma(c)|\tilde{b}\rangle = |c\tilde{b}\rangle. \quad (3.12)$$

The vector $|\tilde{\mathbf{1}}\rangle$ is cyclic in \mathcal{H}_{GNS} , so that all of \mathcal{H}_{GNS} can be obtained from the action of the elements of \mathcal{C} (and its completion in the \mathcal{H}_{GNS} norm), and

$$\omega([a_0]; \rho, \lambda)(c) = \text{Tr}_{\mathcal{H}_{\text{GNS}}} (|\tilde{\mathbf{1}}\rangle \langle \tilde{\mathbf{1}}| \sigma(c)). \quad (3.13)$$

Now, the representation (3.12) is reducible showing that $\omega([a_0]; \rho, \lambda)$ is not pure. We can see this as follows. Since $\mathbf{1} - \mathbb{P} \in \mathcal{N}$,

$$|\tilde{\mathbf{1}}\rangle = |\tilde{\mathbb{P}}\rangle. \quad (3.14)$$

Since $\sigma(\mathcal{C})$ is an $SU(3)$ -singlet, its action does not affect λ . Hence as a state,

$$|\tilde{\mathbb{P}}\rangle \langle \tilde{\mathbb{P}}| \Big|_{\mathcal{C}} = \sum_{\lambda} |\widetilde{\mathbb{P}}_{\lambda}\rangle \langle \widetilde{\mathbb{P}}_{\lambda}|, \quad (3.15)$$

$$|\widetilde{\mathbb{P}}_{\lambda}\rangle := |[a_0]; \rho, \lambda\rangle \langle [a_0]; \rho, \lambda|. \quad (3.16)$$

On each $|\widetilde{\mathbb{P}}_{\lambda}\rangle$, regarded as a cyclic vector, we can build a representation of \mathcal{C} :

$$\sigma(c)|\widetilde{\mathbb{P}}_{\lambda}\rangle = |c\widetilde{\mathbb{P}}_{\lambda}\rangle. \quad (3.17)$$

Thus $|\tilde{\mathbf{1}}\rangle \langle \tilde{\mathbf{1}}|$ restricted to \mathcal{C} is a mixture of $|\rho|$ pure states ($|\rho|$ being the dimension of ρ) and is mixed for $|\rho| \neq 1$.

As discussed elsewhere [8, 9], the decomposition (3.15) is not unique. If $|\mathbb{P}|$ is the rank of \mathbb{P} , $u \in U(|\mathbb{P}|)$ and

$$|\widetilde{\mathbb{P}}'_{\lambda}\rangle = |\widetilde{\mathbb{P}_{\sigma} u_{\sigma\lambda}}\rangle, \quad (3.18)$$

then

$$|\tilde{\mathbf{1}}\rangle \langle \tilde{\mathbf{1}}| \Big|_{\mathcal{C}} = \sum_{\lambda} |\widetilde{\mathbb{P}}'_{\lambda}\rangle \langle \widetilde{\mathbb{P}}'_{\lambda}|. \quad (3.19)$$

This ambiguity introduces ambiguities in entropy.

The group algebra $\mathbb{C}SU(3)$ restricted to the ρ -representation and $\mathbb{C}U(|\mathbb{P}|)$ coincide. Thus the entropy ambiguities emerge from unobserved colour. If colour *were* part of \mathcal{C} , the state (3.3) would remain pure.

The following point is important. Since observables are colour singlets, we can observe only \mathbb{P} and not \mathbb{P}_{λ} or \mathbb{P}'_{λ} . Hence while we can prepare the vector $|\tilde{\mathbb{P}}\rangle$ by observing \mathbb{P} , we cannot prepare $|\widetilde{\mathbb{P}}_{\lambda}\rangle$ or $|\widetilde{\mathbb{P}}'_{\lambda}\rangle$. This with (3.15) shows another way to understand how mixed states arise in QCD.

4 The Matrix model

4.1 The Case of Two Colors: A Review

The basic work leading to this model is that of Narasimhan and Ramadas [3]. They consider the colour group $SU(2)$ and the spatial slice S^3 . We remark that as for fuzzy spheres, we can recover \mathbb{R}^3 from S^3 by suitable limits.

Narasimhan and Ramadas rigorously prove that for $N = 2$, the gauge bundle

$$\mathcal{G}_0^\infty \rightarrow \mathcal{A} \rightarrow \mathcal{A}/\mathcal{G}_0^\infty \quad (4.1)$$

is twisted and does not admit a global section (that is, a gauge fixing). For proving this result, they reduce the problem to one of studying the special left-invariant connections

$$\omega = i(\text{Tr } \tau_i u^{-1} du) M_{ij} \tau_j, \quad (4.2)$$

where τ_i are the Pauli matrices, $u \in SU(2)$ and M is a 3×3 real matrix. The connection on spatial S^3 is obtained by diffeomorphically mapping S^3 onto $SU(2)$ and pulling back ω . The submanifold of such ω is preserved only by the global $SU(2)$ adjoint action

$$\omega \rightarrow v \omega v^{-1}, \quad v \in SU(2), \quad (4.3)$$

or

$$M \rightarrow MR^T \quad (4.4)$$

where R is the $SO(3)$ image of v under the homomorphism $SU(2) \rightarrow SO(3)$. The action of $SO(3)$ on the space \mathcal{M}_0 of 3×3 real matrices of rank ≥ 2 is free and leads to an $SO(3)$ fibration

$$SO(3) \rightarrow \mathcal{M}_0 \rightarrow \mathcal{M}_0/SO(3). \quad (4.5)$$

From this result, they deduce that the gauge bundle is also twisted.

4.2 The Case of Three Colors

We now adapt the preceding discussion to $SU(3)$.

We start with the left-invariant one-form on $SU(3)$,

$$\Omega = \text{Tr} \left(\frac{\lambda_a}{2} u^{-1} du \right) M_{ab} \lambda_b, \quad u \in SU(3), \quad (4.6)$$

where M is a real 8×8 matrix and Tr is in the fundamental representation of $SU(3)$. These M 's parametrize a submanifold of connections \mathcal{A} which captures the essential topology of current interest.

In $SU(3)$, λ_i , $i = 1, 2, 3$, generate an $SU(2) \simeq S^3$ subgroup. We map spatial S^3 diffeomorphically to $SU(2)$,

$$S^3 \ni \vec{x} \rightarrow u(\vec{x}) \in SU(2) \subset SU(3), \quad (4.7)$$

with a distinguished point p having the image $e \in SU(3)$. A convenient choice is the Skyrme ansatz [4]

$$u(\vec{x}) = \begin{pmatrix} \cos \theta(r) + i \tau_i \hat{x}_i \sin \theta(r) & 0 \\ 0 & 1 \end{pmatrix}, \quad \theta(0) = \pi, \quad \theta(\infty) = 0, \quad \vec{x} \in \mathbb{R}^3, \quad r \equiv |\vec{x}| \quad (4.8)$$

Although $\vec{x} \in \mathbb{R}^3$, $\lim_{r \rightarrow \infty} u(\vec{x}) = 1$, so that u gives a mapping from S^3 to $SU(3)$.

Now, if X_i are vector fields on $SU(3)$ representing λ_i for the right action $X_i u = -u\lambda_i/2$, then $[X_i, X_j] = i\epsilon_{ijk}X_k$, and

$$\Omega(X_i) = -M_{ib}\frac{\lambda_b}{2}. \quad (4.9)$$

Thus on identifying spatial vector fields with iX_j , $j = 1, 2, 3$, one has for the vector potentials on the spatial slice,

$$A_j = -iM_{jb}\frac{\lambda_b}{2}. \quad (4.10)$$

Here M has no spatial dependence whereas \mathcal{G}^∞ acting on A_j will introduce such dependence, except at identity (since $U(p) = e$), and will not preserve the form of A_j . This submanifold is thus gauge fixed with respect to \mathcal{G}^∞ (Such gauge fixation is not possible for the space of *all* \mathcal{A} since $\mathcal{A} \neq (\mathcal{A}/\mathcal{G}^\infty) \times \mathcal{G}^\infty$).

But $SU(3)$ of colour acts on A_j . If $h \in SU(3)$,

$$A_j \rightarrow h A_j h^{-1} \quad \text{or} \quad M \rightarrow M(Ad h)^T. \quad (4.11)$$

Remark: For later use, we now show that the action (4.11) is not necessarily free. This result will not be of importance in this paper.

There are four linearly independent vectors in the octet representation of $SU(3)$ which are singlets under hypercharge $Y \propto \lambda_8$, since

$$[\lambda_8, \lambda_i] = [\lambda_8, \lambda_8] = 0, \quad i = 1, 2, 3. \quad (4.12)$$

These correspond to the pions π_i and the eta meson η . Hence if the columns of M are spanned by π_i and η , then

$$M \left(Ad e^{i\alpha\lambda_8} \right) = M. \quad (4.13)$$

It follows that the $Ad SU(3)$ action on M is not free if its rank is ≤ 4 .

But $Ad SU(3)$ does act freely on \mathcal{M}^0 , the space of matrices of rank ≥ 5 . We can see this as follows. Let $M \in \mathcal{M}^0$ and map the columns of M to the 3×3 $SU(3)$ Lie algebra according to

$$M_{i\alpha} \rightarrow M_{i\alpha} \frac{\lambda_\alpha}{2}, \quad i \in [1, 2, \dots, 8]. \quad (4.14)$$

The action $M \rightarrow M (Ad h)^T$ of $Ad SU(3)$ on M is equivalent to its adjoint action on λ_α . So we focus on the vector space spanned by λ_α on which $SU(3)$ acts by conjugation.

Now if an element $h \in SU(3)$ leaves $\xi_\alpha \lambda_\alpha$ and $\eta_\alpha \lambda_\alpha$ invariant under conjugation, it also leaves their product invariant. So the set of such vectors left invariant under $SU(3)$ conjugation forms an algebra. So does their complex linear span. Let \mathcal{F} denotes this complex algebra. This algebra is a $*$ -algebra with the $*$ defined by hermitian conjugation h being unitary. It is then a standard result that \mathcal{F} is the direct sum of full matrix algebras. As \mathcal{F} acts on \mathbb{C}^3 , we can conclude that

$$\mathcal{F} = \bigoplus Mat_{N_i}, \quad \sum N_i = 3. \quad (4.15)$$

We already found an algebra \mathcal{F}' fixed by hypercharge, namely

$$\mathcal{F}' = \left\{ \left(\begin{array}{c|c} m & 0 \\ \hline 0 & c \end{array} \right) \right\}, \quad c \in \mathbb{C}, \quad (4.16)$$

the m being generated by λ_i while \mathbb{C} can be obtained from λ_3^2 and λ_8^2 .

This \mathcal{F}' is maximal if its stabiliser h is not a multiple of 1. For the only bigger \mathcal{F}' is $Mat_3(\mathbb{C})$, and if h commutes with all of $Mat_3(\mathbb{C})$, then h lies in the centre of $SU(3)$. Then $Ad h$ is identity.

We have thus proved that $Ad SU(3)$ acts freely on \mathcal{M}^0 .

Remark: For $N = 2$, and the gauge group $Ad SU(2) = SO(3)$, the matrix M in (4.9) is 3×3 . Narasimhan and Ramadas [3] have remarked that the $SO(3)$ action

$$M \rightarrow Mh^T, \quad h \in SO(3) \quad (4.17)$$

is free if the rank of M is larger than one. Thus \mathcal{M}^0 in this case are real matrices of rank 2 or 3.

4.3 The Matrix Model Bundle is Twisted

Now, the dimension of \mathcal{M} is 64. The dimension of matrices of rank 4 is 32. Hence their codimension is also 32. Furthermore, since \mathcal{M} is contractible, $\pi_j(\mathcal{M}) = 0$ for all j . Hence by Remark 3 to Theorem 6.2 in Narasimhan and Ramadas [3], $\pi_1(\mathcal{M}^0) = 0$. That is enough to show that

$$\mathcal{M}^0 \neq \mathcal{M}^0 / Ad SU(3) \times Ad SU(3) \quad (4.18)$$

since $\pi_1(Ad SU(3)) = \mathbb{Z}_3$.

We thus conclude that the bundle

$$Ad SU(3) \rightarrow \mathcal{M}^0 \rightarrow \mathcal{M}^0 / Ad SU(3) \quad (4.19)$$

captures the $SU(3)$ twist of the exact theory.

Narasimhan and Ramadas in their proof of Theorem 6.2 also show that for $N = 2$, the bundle

$$SO(3) \rightarrow \mathcal{M}^0 \rightarrow \mathcal{M}^0 / SO(3) \quad (4.20)$$

is twisted. This result is important for us as we also consider $N = 2$ explicitly in Sections 5.1 and 6.

4.4 The Hamiltonian for $SU(N)$

Recall that the Yang-Mills action is

$$S = -\frac{1}{2g^2} \int d^4x \operatorname{Tr} F_{\mu\nu} F^{\mu\nu}, \quad \text{with} \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \quad (4.21)$$

Upon rescaling $A \rightarrow gA$, we recover the form used in perturbative QCD.

From the Hamiltonian

$$H = \frac{1}{2} \int d^3x \operatorname{Tr} \left(g^2 E_i E_i - \frac{1}{g^2} F_{ij}^2 \right), \quad E_i = \text{chromoelectric field} \quad (4.22)$$

of (4.21), we can easily write down the Hamiltonian for the reduced matrix model, which we will do in the next section.

As the configuration space variables for the matrix model are $M_{i\alpha}$, it is natural to take the $\frac{d}{dt} M_{i\alpha}$ after Legendre transformation as the conjugate of $M_{i\alpha}$. In QCD, the conjugate to the connection is the chromoelectric field. So we identify this conjugate operator with the matrix model chromoelectric field $E_{i\alpha}$. On quantising the reduced model, these satisfy

$$[M_{i\alpha}, E_{j\beta}] = i\delta_{ij}\delta_{\alpha\beta}. \quad (4.23)$$

5 Matrix Model for $SU(N)$ gauge theory

In the matrix model, A_i plays the role of the vector potential. From its curvature $d\Omega + \Omega \wedge \Omega$, we get

$$(d\Omega + \Omega \wedge \Omega)(iX_i, iX_j) = F_{ij} = i\epsilon_{ijk}M_{k\alpha}\frac{\lambda_\alpha}{2} - if_{\alpha\beta\gamma}M_{i\alpha}M_{j\beta}\frac{\lambda_\gamma}{2}, \quad i = 1, \dots, 3, \quad \alpha = 1, \dots, N^2 - 1, \quad (5.1)$$

where $f_{\alpha\beta\gamma}$ are $SU(N)$ structure constants.

In the reduced matrix model, the term $-(\text{Tr } F_{ij}F_{ij})/2g^2$ plays the role of the potential $V(M)$:

$$V(M) = -\frac{1}{2g^2}(\text{Tr } F_{ij}F_{ij}) \quad (5.2)$$

$$= \frac{1}{2g^2} \left(M_{k\alpha}M_{k\alpha} - \epsilon_{ijk}f_{\alpha\beta\gamma}M_{i\alpha}M_{j\beta}M_{k\gamma} + \frac{1}{2}f_{\alpha_1\beta_1\gamma}f_{\alpha_2\beta_2\gamma}M_{i\alpha_1}M_{j\beta_1}M_{i\alpha_2}M_{j\beta_2} \right) \quad (5.3)$$

The reduced matrix model Hamiltonian is thus

$$H = \frac{1}{R} \left(\frac{g^2 E_{i\alpha} E_{i\alpha}}{2} + V(M) \right) \quad (5.4)$$

We have introduced an overall factor of $1/R$ for dimensional reasons, R having the dimension of length.

Notice that in the limit $g \rightarrow 0$, the potential term $V(M)$ dominates, while the kinetic term dominates in the limit $g \rightarrow \infty$.

As a quantum operator, H is thus given by

$$H = -\frac{g^2}{2} \sum_{i,\alpha} \frac{\partial^2}{\partial M_{i\alpha}^2} + V(M). \quad (5.5)$$

It acts on the Hilbert space of functions ψ_i of M with scalar product

$$(\psi_1, \psi_2) = \int \Pi_{i,\alpha} dM_{i\alpha} \bar{\psi}_1(M) \psi_2(M) \quad (5.6)$$

Previous work on Related Models:

Savvidy has suggested a matrix model for Yang-Mills quantum mechanics [12], which has been explored by many researchers. However, their arguments for arriving at the matrix model differ from ours, as does their potential.

Other investigations of Yang-Mills quantum mechanics involve approximating the gauge field by several $N \times N$ (unitary or hermitian) matrices. The potential V has interesting properties in the large N limit, and several investigations have been carried out by [13–16]. Again, these models differ from our model, in that our model (5.5) is based on a single $3 \times (N^2 - 1)$ real matrix with a kinetic energy term.

5.1 Simplification of Potential and its Extrema: $SU(2)$ Case

Let us specialise to the case of $SU(2)$ gauge theory. Then $f_{\alpha\beta\gamma} = \epsilon_{\alpha\beta\gamma}$. Hence

$$V(M) = \frac{1}{2g^2} \left(\text{Tr } M^T M - 6 \det M + \frac{1}{2} [(\text{Tr } M^T M)^2 - \text{Tr } M^T M M^T M] \right) \quad (5.7)$$

Let us do the singular value decomposition (SVD) of M : $M = RAS^T$, where A is a diagonal matrix with non-negative entries a_i , and R and S are real orthogonal matrices. By applying extra rotations to the right of R or S , we can assume that $a_1 \geq a_2 \geq a_3 \geq 0$. With this decomposition,

$$2g^2V(M) = (a_1^2 + a_2^2 + a_3^2) - 6a_1a_2a_3 + (a_1^2a_2^2 + a_1^2a_3^2 + a_2^2a_3^2) \quad (5.8)$$

Note that under gauge transformations, $M \rightarrow MR^T$ (with $R \in SO(3)$), so $V(M)$ is invariant under gauge transformations.

The potential is zero for $M = 0$, and $M = \mathbb{1}$. These two are gauge-related by a large gauge transformation, because $u(\vec{x})$ is a winding number 1 transformation and for $M = \mathbb{1}$, A is the gauge transform of the zero connection by a winding number 1 transformation.

The minima of V are given by

$$\frac{\partial V}{\partial a_1} = \frac{1}{g^2} (a_1 - 3a_2a_3 + a_1(a_2^2 + a_3^2)) = 0 \quad (5.9)$$

and similar equations from $\partial V/\partial a_2 = 0, \partial V/\partial a_3 = 0$. Symmetry of the equations under $a_i \leftrightarrow a_j$ suggests that all a_i are equal at the extremum. Putting $a_1 = a_2 = a_3 = a$ immediately gives $a = 0, 1/2, 1$ as the extrema.

We can look at the Hessian matrix $\text{Hess} = [\partial^2 V/\partial a_i \partial a_j]$:

$$\text{Hess} = \frac{1}{g^2} \begin{pmatrix} 1 + a_2^2 + a_3^2 & 2a_1a_2 - 3a_3 & 2a_1a_3 - 3a_2 \\ 2a_1a_2 - 3a_3 & 1 + a_3^2 + a_1^2 & 2a_2a_3 - 3a_1 \\ 2a_1a_3 - 3a_2 & 2a_2a_3 - 3a_1 & 1 + a_1^2 + a_2^2 \end{pmatrix} \quad (5.10)$$

This is positive definite at $a_1 = a_2 = a_3 = 0$ with eigenvalues $1/g^2, 1/g^2, 1/g^2$. It is also positive definite at $a_1 = a_2 = a_3 = 1$ with eigenvalues $1/g^2, 4/g^2, 4/g^2$. Even though $M = 0$ and $M = \mathbb{1}$ are related by a (large) gauge transformation, the Hessian has a very different spectrum. The physical consequences of this is unclear to us.

The Hessian at $M = \mathbb{1}/2$ has eigenvalues $-1/2g^2, 5/2g^2, 5/2g^2$. So this extremum is a saddle point. Again, we need to understand the physical interpretation of this saddle point.

Separation of Variables in H :

The quantum mechanical Hamiltonian is given by (5.5) and (5.7). We note that for $N = 2$, its separation of variables into radial coordinates a_i and angular coordinates (R and S) is available in previous work [20, 21].

6 Spectrum of the Hamiltonian

We will work with the Hamiltonian (5.5) and limit ourselves here to qualitative remarks and estimates about its spectrum for $N = 2$ and 3. Detailed work is in progress with S. Digal.

The potential grows quadratically in a_i as $|a_i| \rightarrow \infty$, while it is smooth elsewhere. It follows immediately that the spectrum is gapped as required by colour confinement, and is discrete as well.

The potential resembles that of the anharmonic quartic oscillator. In the latter case, the anharmonic term is known to be a singular perturbation which cannot be treated using perturbation theory [17–19].

We will use variational methods to estimate energy levels. We will be guided by

$$H_0 = \frac{1}{R} \left(\frac{g^2 E_{i\alpha} E_{i\alpha}}{2} + \frac{M_{i\alpha} M_{i\alpha}}{2g^2} \right) \quad (6.1)$$

in our choice of the variational ansatz.

The eigenfunctions of H_0 are of the form $f(M_{i\alpha})e^{-M_{i\alpha}M_{i\alpha}/2g^2}$, where $f(M_{i\alpha})$ are products of Hermite polynomials in $3(N^2 - 1)$ variables $M_{i\alpha}$.

For the variational ansatz for the ground state, we take

$$\Psi_b^0 = A_0 e^{-\frac{b}{2g^2} M_{i\alpha} M_{i\alpha}}, \quad A_0 = \left(\frac{b}{\pi g^2} \right)^{\frac{3}{4}(N^2-1)}, \quad (6.2)$$

and minimise with respect to the parameter b .

We find

$$\langle \Psi_b^0 | H | \Psi_b^0 \rangle \equiv E^0(b, g) = \frac{3}{4R} (N^2 - 1) \left(b + \frac{1}{b} + \frac{g^2 N}{2b^2} \right) \quad (6.3)$$

Minimizing with respect to b gives the variational ground state energy $E_{\min}^0(g)$. It is plotted in Figure 1 as a function of t'Hooft coupling $t = g^2 N$.

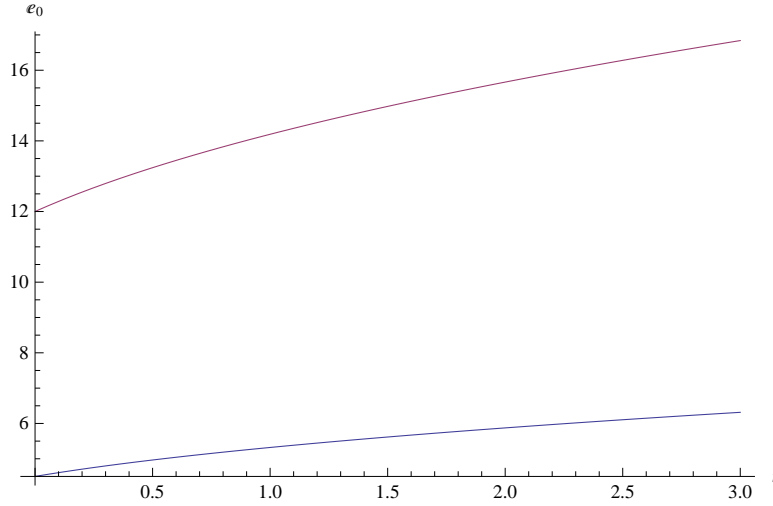


Figure 1: The blue line is for $N = 2$, the red line is for $N = 3$. The energy is in units of $1/R$ and $t = g^2 N$.

Similarly, we can take the ansatz

$$\Psi_b^1 = A_1 M_{i\alpha} e^{-\frac{b}{2g^2} M_{i\alpha} M_{i\alpha}}, \quad A_1 = \frac{2b}{g^2} \left(\frac{b}{\pi g^2} \right)^{\frac{3}{4}(N^2-1)}, \quad (6.4)$$

for the first excited state. This is an impure state because the colour index α is not "soaked up". We then calculate

$$\langle \Psi_b^1 | H | \Psi_b^1 \rangle \equiv E^1(b, g), \quad (6.5)$$

to find

$$E^1(b, g) = \frac{1}{4R} (3N^2 - 1) \left(b + \frac{1}{b} + \frac{g^2 N}{2b^2} \right) + \frac{g^2 N}{4b^2}. \quad (6.6)$$

Its minimum $E_{\min}^1(g)$ is plotted against $g^2 N$ in Figure 2.

Notice that both these trial wave functions are insensitive to the $O(g)$ term in the Hamiltonian. The simplest ansatz that is sensitive to this term is

$$\Phi_{(b,c)}^1 = B_1 (M_{i\alpha} + c \epsilon_{ijk} f_{\alpha\beta\gamma} M_{j\beta} M_{k\gamma}) e^{-\frac{b}{2g^2} M_{i\alpha} M_{i\alpha}}, \quad c \in \mathbb{C}. \quad (6.7)$$

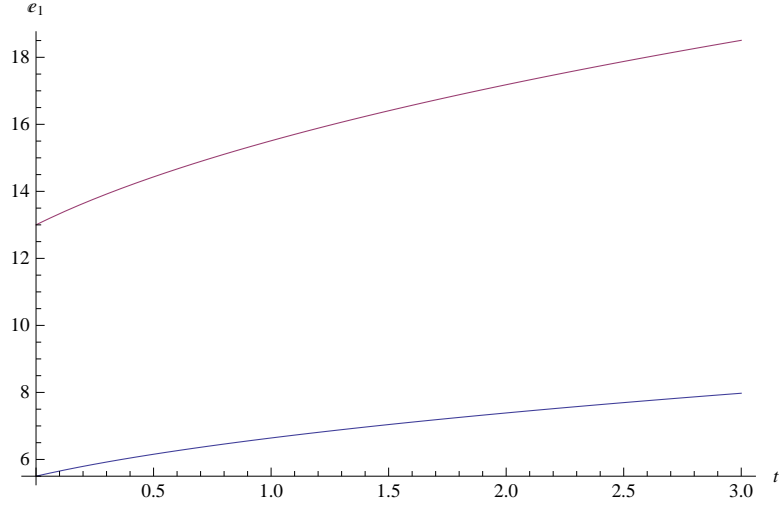


Figure 2: The blue line is for $N = 2$, the red line is for $N = 3$. The energy is in units of $1/R$ and $t = g^2 N$.

This has three variational parameters: c, c^* and b . The variational energy for this ansatz is shown in Figure 3.

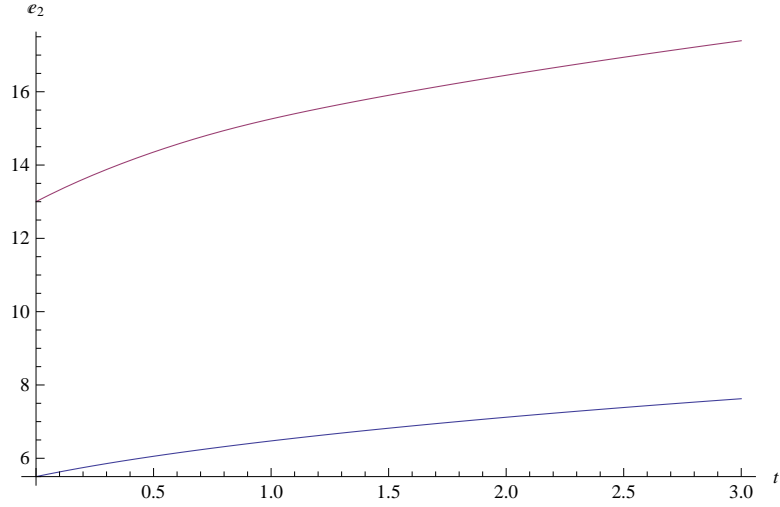


Figure 3: The blue line is for $N = 2$, the red line is for $N = 3$. The energy is in units of $1/R$ and $t = g^2 N$.

Our variational energy estimate is rather crude, and is presented here for representational purposes only. We expect that the variational estimate differs significantly from the true energy for large values of t' Hooft coupling t . Much better numerical estimates may be obtained by taking more sophisticated (or complicated!) variational ansatz for the wavefunctions. We will not do it here.

7 On Mixed States in the Matrix Model

Considerations using \mathcal{A} in sections 2 and 3 were formal, whereas the matrix model for $N = 3$ is that of a particle with 64 degrees of freedom. It is a well-defined quantum mechanical model, which captures the colour twist topology of QCD .

The C^* -algebra $\mathcal{C}(\mathcal{M})$ of the observables are made up of colour singlets. It contains colour singlet functions of M . (More precisely, we consider only bounded operators of this sort). The full $\mathcal{C}(\mathcal{M})$ is generated by such operators.

We can now adapt section 3 to show that coloured states restricted to $\mathcal{C}(M)$ are not pure.

8 Final Remarks

The one definite result we have in the work is the conclusion that coloured states in QCD are mixed. That will affect correlators and partition functions and hence physical predictions. Calculations in this directions have not been done.

In addition, we have developed a matrix model for pure QCD which gives a gapped spectrum and discrete levels for glueballs.

Our present work can be generalised to other gauge groups.

We conclude with a few further remarks on the matrix model.

1. We can couple quarks to A_i by using covariant derivative $\nabla_i = \partial_i + A_i$ in the Dirac operators, this being its only modification in the $A_0 = 0$ gauge.
2. We can construct QCD θ -states as follows. The Chern-Simons 3-form gives the field theory action

$$S_{CS}(A) = \frac{1}{8\pi^2} \int \text{Tr} \left(A \wedge F - \frac{1}{3} A \wedge A \wedge A \right), \quad (8.1)$$

$$= \frac{1}{16\pi^2} \int d^3x \epsilon^{ijk} \text{Tr} \left(A_i(x) F_{jk}(x) - \frac{2}{3} A_i(x) A_j(x) A_k(x) \right) \quad (8.2)$$

which in the matrix model becomes, on using (4.10) and (5.1),

$$S_{CS}(M) = \frac{1}{4} \left[\text{Tr}(M^T M) + \frac{1}{6} \epsilon_{ijk} f_{\alpha\beta\gamma} M_{i\alpha} M_{j\beta} M_{k\gamma} \right]. \quad (8.3)$$

The overall $1/4$ is fixed by requiring that for a pure gauge, where $M = \mathbf{1}_{3 \times 3} \oplus \mathbf{0}_{5 \times 5}$, where $\mathbf{1}_{3 \times 3}$ is in the $SU(2)$ subspace, the RHS becomes the winding number 1. Then under a gauge transformation

$$A \rightarrow hAh^{-1} + hdh^{-1}, \quad (8.4)$$

$S_{CS}(A)$ changes by the winding number $N(h)$ of the map h [4, 22]:

$$S_{CS}(hAh^{-1} + hdh^{-1}) = N(h) + S_{CS}(A), \quad N(h) \in \mathbb{Z}. \quad (8.5)$$

Hence

$$e^{i\theta S_{CS}(hAh^{-1} + hdh^{-1})} = e^{i\theta N(h)} e^{i\theta S_{CS}(A)}. \quad (8.6)$$

Thus given a vector state $|\cdot, \theta = 0\rangle$ for $\theta = 0$, we can get the one $|\cdot, \theta\rangle$ with non-zero θ as follows:

$$|\cdot, \theta\rangle = e^{i\theta S_{CS}(M)} |\cdot, \theta = 0\rangle. \quad (8.7)$$

With this formula, concrete calculations can be done using the Hamiltonian H .

3. We can build multiparticle states for our gluon levels from (4.6) by changing $u(\vec{x})$ to higher winding number maps as in Skyrmin physics [4].
4. That coloured states are impure states have deep implications for the confinement problem. Consider the time-evolution of a pure (and hence colourless) state. Since time evolution in quantum theory is given by a unitary operator, this state will *never* evolve to a coloured state. Thus it is impossible to create a free gluon starting from a colourless state by any Hamiltonian evolution, and in particular by scattering.
5. The mixed coloured states we obtain are convex combinations of pure states. Of these, at most one can be in the domain of the Hamiltonian. Consequently, the mean energy in such mixed states is infinite [23–25]. This is an additional argument in support of the relevance of these mixed states for the confinement problem in Yang-Mills theories.

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